Journal of Statistical Physics, Vol. 52, Nos. 1/2, 1988

One-Dimensional Harmonic Lattice Caricature of Hydrodynamics: Second Approximation

R. L. Dobrushin,[1] **A. Pellegrinotti,**[1,2] **Yu. M. Suhov,**[1,3] **and L. Triolo**[4]

Received December 10, 1987

The long-time behavior of an infinite chain of coupled harmonic oscillators is studied. In addition to a limiting "hydrodynamic" (Euler-type) equation, the "next approximation" is investigated. The corresponding equation is derived.

KEY WORDS: Harmonic oscillators; hydrodynamic limit; second approximation.

1. INTRODUCTION

This paper continues the work[1] devoted to the derivation of a limiting "hydrodynamic" (Euler) equation for the infinite chain of harmonic oscillators. We refer to ref. 1 (where basic notations are taken from) for a detailed discussion of this topic for the model under consideration and to refs. 2–4 for the statement of the problem in a general framework. The "Euler" equation for the harmonic oscillator model is written in terms of a spectral density matrix function (SDMF) profile,

$$\hat{F}(t; x, \theta) = (\hat{F}^{\alpha,\beta}(t; x, \theta), \alpha, \beta = 1, 2), \qquad t, x \in \mathbb{R}^1, \theta \in [-\pi, \pi)$$

[1] Institute for Problems of Information Transmission, USSR Academy of Sciences, 101447 GSP-4, Moscow, USSR.

[2] Dipartimento di Matematica, Universita' di Roma "La Sapienza", 00100 Rome, Italy, and GNFM-CNR.

[3] Research Center BiBoS, Universitat Bielefeld, D-4800 Bielefeld, Federal Republic of Germany.

[4] Dipartimento di Metodi e Modelli Matematici, Universita' di Roma "La Sapienza", 00100 Rome, Italy, and GNFM-CNR.

which describes the "macroscopic" time evolution of local equilibrium parameters. It reads

$$\frac{\partial}{\partial t}\hat{F}(t;x,\theta)=A(\theta)\frac{\partial}{\partial x}\hat{F}(t;x,\theta), \qquad \theta\in[-\pi,\pi), \quad t,x\in\mathbb{R}^1 \tag{1.1}$$

where

$$A(\theta)=i\omega'(\theta)\begin{pmatrix}0 & -1/\omega(\theta)\\ \omega(\theta) & 0\end{pmatrix} \tag{1.2}$$

and

$$\omega(\theta)=\left[\sum_{k\in\mathbb{Z}}V(k)\exp(ik\theta)\right]^{1/2}, \qquad \theta\in[-\pi,\pi)$$

$V(\cdot)$ is the harmonic interaction sequence. The precise definition of the SDMF is (see Theorem 3.1 below)

$$1/2\pi\int_{-\pi}^{\pi}d\theta\, e^{-ik\theta}\hat{F}^{\alpha,\beta}(t;x,\theta)=\lim_{\varepsilon\to 0}\langle y^{\alpha}_{[\varepsilon^{-1}x]}(\varepsilon^{-1}t)\, y^{\beta}_{[\varepsilon^{-1}x]+k}(\varepsilon^{-1}t)\rangle_{p^{\varepsilon}}$$

where $y^1_j=q_j$ is the position and $y^2_j=p_j$ the momentum of the jth oscillator.

The main goal of this paper is to derive an equation for the next correction (in various senses) to the limiting Euler equation.

Our interest in this problem comes from a widespead belief (which was presumably initiated by a remark of Morrey [5]) telling that the Navier-Stokes equations appear when one takes into account, in addition to the limiting formulas, the first order correction in ε, the scaling space time parameter. It turns out that a "simple" version of the "corrected" equation for the harmonic oscillator chain is of the form

$$\frac{\partial}{\partial t}\hat{F}(t;x,\theta)=A(\theta)\frac{\partial}{\partial x}\hat{F}(t;x,\theta)+\varepsilon B(\theta)\frac{\partial^2}{\partial x^2}\hat{F}(t;x,\theta)\ \theta\in[-\pi,\pi),\ t,x\in\mathbb{R}^1 \tag{1.3}$$

where

$$B(\theta)=i\omega''(\theta)\begin{pmatrix}0 & -1/\omega(\theta)\\ \omega(\theta) & 0\end{pmatrix} \tag{1.4}$$

The precise formulation of this assertion is given by Theorem 3.1. We must say that this theorem, in its present form, establishes a "correcting" property of Eq. (1.3) (w.r.t. Eq. (1.1)) only in a weak sense: Eq. (1.3) provides a good approximation for the harmonic oscillator evolution on larger time intervals. However, we are yet not able to prove (although it looks reasonable) that Eq. (1.3) takes into account *all* terms of the order ε, i.e. to prove that

$$\left| 1/2\pi \int_{-\pi}^{\pi} d\theta e^{-ik\theta} \hat{F}^{\alpha,\beta}(t; x, \theta) - \langle y^{\alpha}_{[\varepsilon^{-1}x]}(\varepsilon^{-1}t)\, y^{\beta}_{[\varepsilon^{-1}x]+k}(\varepsilon^{-1}t) \rangle_{p^{\varepsilon}} \right| \sim 0(\varepsilon)$$

It could be appealing to associate the additional term in the r.h.s. of (1.3) with a "viscosity" of a "medium" composed by harmonic oscillators. However, we prefer to be more reserved about this point of view. We notice that "higher order" corrections to the Euler equation also may be derived. They have a structure similar to (1.3) and are established by means of similar arguments.

The paper is organized as follows. After a short preliminary section we present in Section 3 a simplified version for deriving the Euler equation (1.1).[5] The next approximation is discussed in Section 4.

2. PRELIMINARIES

Let $\chi = (\mathbb{R}^1 \times \mathbb{R}^1)^{\mathbb{Z}}$ be the phase space of the one-dimensional classical spin system on $\mathbb{Z}$. The harmonic oscillator system is described by the following formal Hamiltonian:

$$H(\mathbf{x}) = \sum_{i \in \mathbb{Z}} \left[p_i^2/2 + \sum_{k \in \mathbb{Z}} V(|i-k|)\, q_i q_k \right] \tag{2.1}$$

(the symbol $\mathbb{Z}$ in the summation will be omitted).

We assume that the potential has the same properties as in ref. 1. It is well known that the infinite system of motion equations associated with the Hamiltonian (2.1) has a unique solution for initial conditions from a subset χ' of χ that is large enough to be the support of interesting states (probability measures on χ). Fof details see refs. 6 and 7.

Given $\mathbf{x} = (q_1, p_1) \in \chi'$, we can write the solution of the motion equations in the standard way:

$$q_k(t) = \sum_l \left[U^{1,1}_{k-l}(t)\, q_l + U^{1,2}_{k-l}(t)\, p_l \right] \tag{2.2a}$$

$$p_k(t) = \sum_l \left[U^{2,1}_{k-l}(t)\, q_1 + U^{1,1}_{k-l}(t)\, p_l \right] \tag{2.2b}$$

[5] This is based on a remark due to J. Fritz.

where

$$U_k^{1,1}(t) = 1/2\pi \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \cos[\omega(\theta)t] \tag{2.3a}$$

$$U_k^{1,2}(t) = 1/2\pi \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \frac{\sin[\omega(\theta)t]}{\omega(\theta)} \tag{2.3b}$$

$$U_k^{2,1}(t) = -1/2\pi \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \omega(\theta) \sin[\omega(\theta)t] \tag{2.3c}$$

and

$$\omega(\theta) = \left[\sum_k e^{ik\theta} V(|k|)\right]^{1/2} \tag{2.4}$$

3. THE EULER LIMIT

In this section we give a simpler derivation of the Euler limit for the harmonic oscillator system.

Let a family of 2×2 matrices $\hat{F}(x,\theta) = \{\hat{F}^{\alpha,\beta}(x,\theta), \alpha, \beta = 1, 2\}$, $x \in \mathbb{R}^1$, $\theta \in [-\pi, \pi)$ (an initial SDMF profile), be given with the following properties[1]:

A. For every fixed $x \in \mathbb{R}^1$ and $\alpha, \beta = 1, 2$ the function $\theta \to \hat{F}^{\alpha,\beta}(x,\theta)$ is bounded on $[-\pi, \pi)$ and the inverse Fourier transform

$$F_k^{\alpha,\beta}(x) = 1/2\pi \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \hat{F}^{\alpha,\beta}(x,\theta)$$

satisfies the bound

$$\sup_{x \in \mathbb{R}} |F_k^{\alpha,\beta}(x)| \leqslant a_1 \exp(-a_0 |k|), \qquad k \in \mathbb{Z}^1$$

where a_0, a_1 are positive constants.

B. For every fixed $x \in \mathbb{R}^1$ the diagonals $\hat{F}^{1,1}(x,\cdot)$ and $\hat{F}^{2,2}(x,\cdot)$ are nonnegative even functions and the off-diagonals $\hat{F}^{1,2}(x,\cdot)$ and $\hat{F}^{2,1}(x,\cdot)$ obey

$$\hat{F}^{1,2}(x,-\theta) = \overline{\hat{F}^{1,2}(x,\theta)} = \overline{\hat{F}^{2,1}(x,-\theta)} = \hat{F}^{2,1}(x,\theta), \qquad \theta \in [-\pi, \pi)$$

C. For fixed $x \in \mathbb{R}^1$ and $\theta \in [-\pi, \pi)$ the matrix $\hat{F}(x,\theta)$ is positive semidefinite.

D. For every $\theta \in [-\pi, \pi)$ the functions $\hat{F}^{\alpha,\beta}(\cdot, \theta)$, $\alpha, \beta = 1, 2$ are C^1 and the function

$$x \to \sup_{\theta \in [-\pi, \pi)} \left| \frac{\partial}{\partial x} \hat{F}^{\alpha,\beta}(x, \theta) \right|$$

is bounded uniformly on finite intervals.

Let $\{P^\varepsilon\}$, $\varepsilon > 0$, be a family of states with the following properties:

1. $\langle y_j^\alpha \rangle_{P^\varepsilon} = 0, \qquad j \in \mathbb{Z}, \quad \varepsilon > 0, \quad \alpha = 1, 2$
2. $|\langle y_j^\alpha y_k^\beta \rangle_{P^\varepsilon} - F_{j-k}^{\alpha,\beta}(\varepsilon j)|$
$$\leqslant \min[b_1 \exp(-a\,|j-k|), \varepsilon b_0\,|j-k|], \qquad \alpha, \beta = 1, 2$$

where b_0 and b_1 are suitable positive constants.

3. $|\langle y_j^\alpha y_k^\beta \rangle_{P^\varepsilon}| < \psi(|j-k|), \qquad \sum_{h>0} h^\gamma \psi(h) < \infty \qquad \text{with} \quad \gamma \geqslant 2$

where ψ is monotonic.

Here and below we use the notation

$$y_j^1 = q_j, \qquad y_j^2 = p_j, \qquad j \in \mathbb{Z}$$

In the Euler regime we study the limits

$$F_l^{\alpha,\beta}(t; x) = \lim_{\varepsilon \to 0} \langle y_{[\varepsilon^{-1}x]}^\alpha(\varepsilon^{-1}t)\, y_{[\varepsilon^{-1}x]+1}^\beta(\varepsilon^{-1}t) \rangle_{P^\varepsilon}, \qquad \alpha, \beta = 1, 2 \tag{3.1}$$

which are related to the locally invariant quantities.

The following theorem describing the Euler limit was proven in ref. 1:

Theorem 3.1. Under conditions A–D and 1–3 the limits (3.1) exist and for t different from zero are given by

$$F_k^{\alpha,\beta}(t; x) = 1/2\pi \int_{-\pi}^{\pi} d\theta\, e^{-ik\theta} \hat{F}^{\alpha,\beta}(t; x, \theta), \qquad \alpha, \beta = 1, 2$$

where

$$\begin{aligned}
&\hat{F}^{1,1}(t; x, \theta) \\
&\quad = (1/4)\{\hat{F}^{1,1}(x + \omega'(\theta)\, t, \theta) + \hat{F}^{1,1}(x - \omega'(\theta)\, t, \theta)\} \\
&\qquad + [1/4\omega(\theta)^2]\{\hat{F}^{2,2}(x + \omega'(\theta)\, t, \theta) + \hat{F}^{2,2}(x - \omega'(\theta)\, t, \theta)\} \\
&\qquad + [1/2\omega(\theta)]\{\operatorname{Im} \hat{F}^{1,2}(x + \omega'(\theta)\, t, \theta) - \operatorname{Im} \hat{F}^{1,2}(x - \omega'(\theta)\, t, \theta)\}
\end{aligned}$$

$$\hat{F}^{1,2}(t; x, \theta)$$
$$= [i\omega(\theta)/4]\{\hat{F}^{1,1}(x+\omega'(\theta)\, t, \theta) - \hat{F}^{1,1}(x-\omega'(\theta)\, t, \theta)\}$$
$$+ [i/4\omega(\theta)]\{\hat{F}^{2,2}(x+\omega'(\theta)\, t, \theta) - \hat{F}^{2,2}(x-\omega'(\theta)\, t, \theta)\}$$
$$+ (i/2)\{\operatorname{Im} \hat{F}^{1,2}(x+\omega'(\theta)\, t, \theta) + \operatorname{Im} \hat{F}^{1,2}(x-\omega'(\theta)\, t, \theta)\}$$
$$\hat{F}^{2,1}(t; x, \theta) = -\hat{F}^{1,2}(t; x, \theta)$$
$$\hat{F}^{2,2}(t; x, \theta) = \omega(\theta)^2\, \hat{F}^{1,1}(t; x, \theta)$$

The matrix function

$$\hat{F}(t; x, \theta) = (\hat{F}^{\alpha,\beta}(t; x, \theta), \alpha, \beta = 1, 2), \qquad t, x \in \mathbb{R}^1, \quad \theta \in [-\pi, \pi)$$

satisfies Eq. (1.1).

It is useful to notice that for any $t \in \mathbb{R}^1$ the family of 2×2 matrices

$$\hat{F}(t; x, \theta) = (\hat{F}^{\alpha,\beta}(t; x, \theta), \alpha, \beta = 1, 2), \qquad x \in \mathbb{R}^1, \quad \theta \in [-\pi, \pi)$$

satisfies conditions B and C (and condition D, too). Hence, one can consider this family as the SDMF profile at the macroscopic time moment t.

Examples of families of states that obey 1–3 are discussed in ref. 1.

In this section we give a short proof of this theorem.

Proof. Fix $x \in \mathbb{R}^1$ and $l \in \mathbb{Z}$. As in ref. 1, we consider Eq. (3.1) for the case $\alpha = \beta = 1$ and evaluate the contribution given by the $(q\text{–}q)$ covariance only. We have to study the limit of the sum

$$\sum_{n,n'} U_n^{1,1}(\varepsilon^{-1}t)\, U_{n'+1}^{1,1}(\varepsilon^{-1}t) \langle q_{[\varepsilon^{-1}x]-n}\, q_{[\varepsilon^{-1}x]-n'} \rangle_{P^\varepsilon} \tag{3.2}$$

At the first stage, as in ref. 1, one passes from (3.2) to

$$\sum_{n,n'} U_n^{1,1}(\varepsilon^{-1}t)\, U_{n'+1}^{1,1}(\varepsilon^{-1}t)\, \hat{F}_{n'-n}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n))$$

and then to the Fourier transform

$$1/4\pi^2 \sum_n \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\, \varepsilon^{-1}t]$$
$$\times \int_{-\pi}^{\pi} d\varphi\, e^{-in\varphi} \cos[\omega(\theta-\varphi)\, \varepsilon^{-1}t]\, \hat{F}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n), \theta) \tag{3.3}$$

Given $x \in \mathbb{R}^1$, we introduce a space cutoff for the function $\hat{F}^{1,1}(y, \theta)$, i.e., consider a function $\hat{F}_0^{1,1}(y, \theta) = \hat{F}_{x,0}^{1,1}(y, \theta)$, $y \in \mathbb{R}^1$, $\theta \in [-\pi, \pi)$, such that

(i) $\hat{F}_0^{1,1}(y, \theta) = \hat{F}^{1,1}(y, \theta), \quad y \in [x - c_0 t, x + c_0 t], \quad \alpha, \beta = 1, 2$

(ii) $\hat{F}_0^{1,1}(y, \theta) = 0, \quad y \notin [x - c_0 t - c_1, x + c_0 t - c_1], \quad \theta \in [-\pi, \pi)$

(iii) $\hat{F}_0^{1,1}(\cdot, \theta) \in C_0^\infty, \quad \forall \theta \in [-\pi, \pi)$

(iv) $$\sup_{y \in \mathbb{R}} \sup_{\theta \in [-\pi,\pi)} \max(|\hat{F}_0^{1,1}(y, \theta)|, |(\partial/\partial x)\, \hat{F}_0^{\alpha,\beta}(x, \theta)|, |(\partial^2/\partial x^2)\, \hat{F}_0^{\alpha,\beta}(x, \theta)|) < \infty$$

where $c_0 > \max |\omega'(\theta)|$ and $c_1 > 0$ are constants.

In the next stage of the proof we replace the term $\hat{F}^{1,1}(\varepsilon([\varepsilon^{-1}x] - n), \theta)$ in (3.3) by the approximate value $\hat{F}_0^{1,1}(\varepsilon([\varepsilon^{-1}x] - n), \theta)$. The difference

$$1/4\pi^2 \sum_n \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\, \varepsilon^{-1} t] \int_{-\pi}^{\pi} d\varphi\, e^{in\varphi} \cos[\omega(\theta + \varphi)\, \varepsilon^{-1} t] \times [\hat{F}^{1,1}(\varepsilon([\varepsilon^{-1}x] - n), \theta) - \hat{F}_0^{1,1}(\varepsilon([\varepsilon^{-1}x] - n), \theta)]$$

contains nonzero addends for $|n| > c_0 \varepsilon^{-1} t$ only and goes to zero due to rapid decreasing of the integral

$$\int_{-\pi}^{\pi} d\varphi\, e^{in\varphi} \cos[\omega(\theta + \varphi)\, \varepsilon^{-1} t]$$

for such values of n [see ref. 1, Lemma 2.6(iii)].

Now we use the Poisson formula [see, e.g., ref. 8, Chapter XIX, formula (5.2)]

$$\sum_{k=-\infty}^{+\infty} \hat{f}(\xi + 2k\lambda) = \pi/\lambda \sum_{n=-\infty}^{+\infty} f(n\pi/\lambda)\, e^{i(n\pi/\lambda)\xi}, \quad f \in C_0^\infty, \quad \xi, \lambda \in \mathbb{R}^1, \quad \lambda \neq 0 \tag{3.4}$$

where $\hat{f}$ is the Fourier transform of f.

Denoting

$$\hat{F}_{0,\xi}^{1,1}(-y, \theta) = \hat{F}_0^{1,1}(\xi - y, \theta) \tag{3.5}$$

we get

$$\sum_n e^{in\varphi} \hat{F}_0^{1,1}(\varepsilon([\varepsilon^{-1}x] - n), \theta) = \varepsilon^{-1} \sum_k {}^\frown\hat{F}_{0,\varepsilon[\varepsilon^{-1}x]}^{1,1}(\varepsilon^{-1}(\varphi + 2k\pi), \theta) \tag{3.5'}$$

where ${}^\frown\hat{F}_{0,\varepsilon[\varepsilon^{-1}x]}^{1,1}(v, \theta)$, $v \in \mathbb{R}^1$, $\theta \in [-\pi, \pi)$, denotes the "spatial" Fourier transform of $\hat{F}_{0,\varepsilon[\varepsilon^{-1}x]}^{1,1}(-y, \theta)$, i.e., the integral

$$\int dy\, e^{ivy} \hat{F}_{0,\varepsilon[\varepsilon^{-1}x]}^{1,1}(-y, \theta)$$

Substituting (3.5′) into the expression under consideration, we obtain

$$1/4\pi^2 \sum_n \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\,\varepsilon^{-1}t]$$

$$\times \int_{-\pi}^{\pi} d\varphi\, e^{in\varphi} \cos[\omega(\theta-\varphi)\,\varepsilon^{-1}t]\, \hat{F}_0^{1,1}(\varepsilon([\varepsilon^{-1}x]-n), \theta)$$

$$= 1/4\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\,\varepsilon^{-1}t] \int_{-\pi}^{\pi} d\varphi \cos[\omega(\theta+\varphi)\,\varepsilon^{-1}t]$$

$$\times \varepsilon^{-1} \sum_k {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}(\varphi+2k\pi), \theta) \tag{3.6}$$

Using the fact that $\hat{F}_0^{1,1}(\cdot, \theta)$ belongs to C_0^∞, one concludes that the contribution of the sum

$$\varepsilon^{-1} \sum_{k\neq 0} {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}(\varphi+2k\pi), \theta)$$

is going to zero as $\varepsilon \to 0$. In fact, this sum does not exceed, in absolute value, the quantity

$$\begin{aligned} &c_3\varepsilon^{-1}\{\exp[-c_2\varepsilon^{-1}(2\pi+\varphi)] + \exp[-c_2\varepsilon^{-1}(2\pi-\varphi)]\} \\ &\quad \leqslant 2c_3\varepsilon^{-1}\exp(-c_2\varepsilon^{-1}\pi) \end{aligned}$$

where c_2 and c_3 are positive constants.

So we have to compute the limit of the term corresponding to $k=0$,

$$1/4\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\,\varepsilon^{-1}t]\,\varepsilon^{-1}$$

$$\times \int_{-\pi}^{\pi} d\varphi \cos[\omega(\theta-\varphi)\,\varepsilon^{-1}t]\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi, \theta)$$

$$= 1/4\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta}(e^{i\omega(\theta)\varepsilon^{-1}t})/2$$

$$\times \int_{-\pi}^{\pi} d\varphi\, (e^{i\omega(\theta-\varphi)\varepsilon^{-1}t} + e^{-i\omega(\theta-\varphi)\varepsilon^{-1}t})/2$$

$$\times \varepsilon^{-1}\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi, \theta) \tag{3.7}$$

We write the rhs of (3.7) as the sum of four terms corresponding to the products $\exp[\pm i\omega(\theta)\varepsilon^{-1}t]\exp[\pm i\omega(\theta-\varphi)\varepsilon^{-1}t]$. All these terms are investigated in the same way. As a result, the nonzero contribution is just given by the two terms that correspond to different signs of the exponents. For definiteness, we consider the integral

$$1/16\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} e^{-i\omega(\theta)\varepsilon^{-1}t} \int_{-\pi}^{\pi} d\varphi\, e^{i\omega(\theta+\varphi)\varepsilon^{-1}t} \times \varepsilon^{-1}\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi, \theta) \tag{3.8}$$

(the integral with the opposite signs of the exponents is treated in the same way).

We write the usual Taylor sum representation for $\omega(\theta+\varphi)$,

$$\omega(\theta+\varphi)=\omega(\theta)+\omega'(\theta)\,\varphi+h(\theta,\varphi)\,\varphi^2$$

and arrive at the following equality:

$$(3.8)=1/16\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \int_{-\pi}^{\pi} d\varphi \exp[+i\omega'(\theta)\,\varphi\varepsilon^{-1}t+ih(\theta,\varphi)\,\varphi^2\varepsilon^{-1}t] \times \varepsilon^{-1}\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi, \theta) \tag{3.9}$$

After the change of variables $\varepsilon^{-1}\varphi=z$, the rhs of (3.9) takes the form

$$1/16\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \int_{-\varepsilon^{-1}\pi}^{\varepsilon^{-1}\pi} dz \exp[+i\omega'(\theta)\,zt+ih(\theta,\varepsilon z)\,z^2\varepsilon t]\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(z,\theta) \tag{3.10}$$

In fact, since $\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}$ has compact support, we can restrict the integral in (3.10) to a finite region (independent of ε).

Writing

$$\begin{aligned}
&1/2\pi \int_{-u}^{u} dz \exp[+i\omega'(\theta)\,zt+ih(\theta,\varepsilon z)\,z^2\varepsilon t]\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(z,\theta)\\
&\quad= 1/2\pi \int_{-u}^{u} dz \exp[+i\omega'(\theta)\,zt]\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(z,\theta)\\
&\qquad+\frac{1}{2\pi}\int_{-u}^{u} dz \exp[i\omega'(\theta)\,zt]\,\{\exp[ih(\theta,\varepsilon z)\,z^2\varepsilon t]-1\}\, {}^{\wedge}\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(z,\theta)
\end{aligned} \tag{3.11}$$

we observe that the last integral goes to zero for any fixed u. On the other hand, the first integral on the rhs of (3.11) asymptotically equals

$$\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(\omega'(\theta)\,t,\theta)=\hat{F}^{1,1}_0(\varepsilon[\varepsilon^{-1}x]-\omega'(\theta)\,t,\theta) \tag{3.12}$$

This goes to $\hat{F}_0^{1,1}(x-\omega'(\theta)\,t, \theta)$ as $\varepsilon \to 0$. The other integral coming from (3.7) tends to $\hat{F}_0^{1,1}(x+\omega'(\theta)\,t, \theta)$.

The last step is to remove the space cutoff and pass from $\hat{F}_0^{1,1}$ to $\hat{F}^{1,1}$. This may be done because the Fourier transform $\hat{\ }\hat{F}^{1,1}_{0,\varepsilon[\varepsilon^{-1}x]}(z, \theta)$ is rapidly decreasing. Thereby we get the result.

4. THE NEXT APPROXIMATION

In this section we study the further approximation in the hydrodynamic behavior of the system. More precisely, we consider the asymptotic behavior, as $\varepsilon \to 0$, of the quantity

$$\langle y^{\alpha}_{[\varepsilon^{-1}x]}(\varepsilon^{-2}t)\, y^{\beta}_{[\varepsilon^{-1}x]+1}(\varepsilon^{-2}t)\rangle_{P^{\varepsilon}}, \qquad \alpha, \beta = 1, 2 \tag{4.1}$$

where $\{P^{\varepsilon}, \varepsilon > 0\}$ is a family of states satisfying the conditions 1–3 of Section 3. This means that we are looking at the system in a macroscopic time $\varepsilon^{-1}t$.

We shall need a new assumption D′ on the initial SDMF profile $\{\hat{F}(x, \cdot), x \in \mathbb{R}^1\}$ which is somewhat stronger than that of D (see Section 3):

D. For every $\theta \in [-\pi, \pi)$, the functions $\hat{F}^{\alpha\beta}(\cdot, \theta)$ $\alpha, \beta = 1, 2$, are C^1 and admit the representation

$$\hat{F}^{\alpha\beta}(x, \theta) = 1/2\pi \int_{\mathbb{R}} \hat{\mu}^{\alpha\beta}(\theta, ds)\, e^{-isk} \tag{4.2}$$

where $\hat{\mu}^{\alpha\beta}(\theta, ds)$, $\theta \in [-\pi, \pi)$, is a (complex) Borel measure on $\mathbb{R}$ depending on the parameter $\theta \in [-\pi, \pi)$ and satisfying the condition

$$\sup_{\theta} \operatorname{Var}(\hat{\mu}^{\alpha\beta}(\theta, \cdot)|_{(-s-1, -s] \cup [s, s+1)}) \leqslant a_2(1+s)^{-2-\delta}, \qquad s > 0$$

with constants $a_2 > 0$ and $\delta > 2/3$.

The result of this section (and the main result of this paper) is the following theorem.

Theorem 4.1. Under conditions A–C and 1–3 of Section 3 and condition D′ of this section the following equality holds:

$$\lim_{\varepsilon \to 0} [\langle y^{\alpha}_{[\varepsilon^{-1}x]}(\varepsilon^{-2}t)\, y^{\beta}_{[\varepsilon^{-1}x]+l}(\varepsilon^{-2}t)\rangle_{P^{\varepsilon}} - F^{\alpha\beta}_{(\varepsilon)}(t; x; l)] = 0 \tag{4.3}$$

where the functions $\{F^{\alpha\beta}_{(\varepsilon)}(t; x; l);\ \alpha, \beta = 1, 2\}$ are given by

$$F^{\alpha\beta}_{(\varepsilon)}(t; x; l) = 1/2\pi \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \hat{F}^{\alpha\beta}_{(\varepsilon)}(t; x; \theta)$$

with

$$\begin{aligned}
&\hat{F}^{1,1}_{(\varepsilon)}(t; x; \theta) \\
&\quad = 1/4[2\pi\omega''(\theta)t]^{-1/2} \int dz(\{\exp[-iz^2/2\omega''(\theta)t]\sqrt{i}\} \\
&\qquad \times \{\hat{F}^{1,1}(x+z+\omega'(\theta)\varepsilon^{-1}t, \theta) + \omega(\theta)^{-2}\hat{F}^{2,2}(x+z+\omega'(\theta)\varepsilon^{-1}t, \theta) \\
&\qquad + 2[\omega(\theta)]^{-1}\operatorname{Im}\hat{F}^{1,2}(x+z+\omega'(\theta)\varepsilon^{-1}t, \theta)\} \\
&\qquad + (1/\sqrt{i})\exp[iz^2/2\omega''(\theta)t]\{\hat{F}^{1,1}(x+z-\omega'(\theta)\varepsilon^{-1}t, \theta) \\
&\qquad + \omega(\theta)^{-2}\hat{F}^{2,2}(x+z-\omega'(\theta)\varepsilon^{-1}t, \theta) \\
&\qquad - 2[\omega(\theta)]^{-1}\operatorname{Im}\hat{F}^{1,2}(x+z-\omega'(\theta)\varepsilon^{-1}t, \theta)\})
\end{aligned}$$

$$\begin{aligned}
&\hat{F}^{1,2}_{(\varepsilon)}(t; x; \theta) \\
&\quad = 1/4[2\pi\omega''(\theta)t]^{-1/2} \int dz(\{\exp[-iz^2/2\omega''(\theta)t]\sqrt{i}\} \\
&\qquad \times \{i\omega(\theta)\hat{F}^{1,1}(x+z+\omega'(\theta)\varepsilon^{-1}t, \theta) + i\omega(\theta)^{-1} \\
&\qquad \times \hat{F}^{2,2}(x+z+\omega'(\theta)\varepsilon^{-1}t, \theta) \\
&\qquad + 2\operatorname{Im}\hat{F}^{1,2}(x+z+\omega'(\theta)\varepsilon^{-1}t, \theta)\} \\
&\qquad + (1/\sqrt{i})\exp[iz^2/2\omega''(\theta)t]\{-i\omega(\theta)\hat{F}^{1,1}(x+z-\omega'(\theta)\varepsilon^{-1}t, \theta) \\
&\qquad - i\omega(\theta)^{-1}\hat{F}^{2,2}(x+z-\omega'(\theta)\varepsilon^{-1}t, \theta) \\
&\qquad + 2\operatorname{Im}\hat{F}^{1,2}(x+z-\omega'(\theta)\varepsilon^{-1}t, \theta)\})
\end{aligned}$$

$$\hat{F}^{2,1}_{(\varepsilon)}(t; x; \theta) = -\hat{F}^{1,2}_{(\varepsilon)}(t; x; \theta)$$

$$\hat{F}^{2,2}_{(\varepsilon)}(t; x; \theta) = \omega(\theta)^2\,\hat{F}^{1,1}_{(\varepsilon)}(t; x; \theta)$$

The matrix function $\hat{F}^{(\varepsilon)}(t; x; \theta) := \hat{F}_{(\varepsilon)}(\varepsilon t; x; \theta)$ satisfies Eq. (1.3). We do not claim that the family of matrices $\{\hat{F}_{(\varepsilon)}(t; x; \theta),\ t, x \in \mathbb{R}^1, \theta \in [-\pi, \pi)\}$ satisfies conditions A–D of Section 3: it seems that in a general situation conditions B and C are violated.

Proof. As above, let $t > 0$. Fix $x \in \mathbb{R}$ and $l \in \mathbb{Z}$. We again consider the relation (4.1) for $\alpha = \beta = 1$ and restrict ourselves to evaluating the contribution given by the $(q\text{–}q)$ covariance. We now have to study the limit of the sum

$$\sum_{n,n'} U^{1,1}_n(\varepsilon^{-2}t)\, U^{1,1}_{n'+1}(\varepsilon^{-2}t) \langle q_{[\varepsilon^{-1}x]-n}\, q_{[\varepsilon^{-1}x]-n'}\rangle_{P^\varepsilon} \tag{4.4}$$

The first step is to pass from (4.4) to

$$\sum_{n,n'} U^{1,1}_n(\varepsilon^{-2}t)\, U^{1,1}_{n'+1}(\varepsilon^{-2}t)\, F^{1,1}_{n'-n}(\varepsilon([\varepsilon^{-1}x]-n))$$

and then to the Fourier transform

$$\sum_{n\in\mathbb{Z}} 1/4\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\,\varepsilon^{-2}t]$$
$$\times \int_{-\pi}^{\pi} d\varphi\, e^{-in\varphi} \cos[\omega(\theta-\varphi)\,\varepsilon^{-2}t]\, \hat{F}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n),\theta) \quad (4.5)$$

[cf. (3.3)]. The difference

$$\sum_{n,n'} U_n^{1,1}(\varepsilon^{-2}t)\, U_{n'+1}^{1,1}(\varepsilon^{-2}t)[\langle q_{[\varepsilon^{-1}x]-n} q_{[\varepsilon^{-1}x]-n'}\rangle_{P^\varepsilon} - F_{n'-n}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n))]$$

is estimated by using Lemma 2.6 from ref. 1 and condition 2 (see Section 3). It is less than

$$c_4 \sum_n |U_n^{1,1}(\varepsilon^{-2}t)| \Bigg(\sum_{n':|n-n'|<-\log\varepsilon} \varepsilon\,|n-n'|$$
$$+ \sum_{n':|n-n'|\geqslant -\log\varepsilon} e^{-a|n-n'|} \Bigg) |U_{n'+1}^{1,1}(\varepsilon^{-2}t)|$$
$$\leqslant c_6 \varepsilon^{2/3} \sum_n |U_n^{1,1}(\varepsilon^{-2}t)|\; [\varepsilon(\log\varepsilon)^2 + c_5\varepsilon]$$
$$\leqslant c_7 \varepsilon^{2/3} \varepsilon^{-4/3} [\varepsilon(\log\varepsilon)^2 + c_5\varepsilon]$$

where $c_4,\dots, c_7$ are positive constants. This goes to zero as $\varepsilon \to 0$. We used here the estimates from Lemma 2.6 in ref. 1.

In the next step we replace the function $\hat{F}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n),\theta)$ in (4.4) with $\hat{F}_{(\varepsilon)}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n),\theta)$, where the function $\hat{F}_{(\varepsilon)}^{1,1}(y,\theta) = \hat{F}_{(\varepsilon),x}^{1,1}(y,\theta)$ is given by

$$\hat{F}_{(\varepsilon)}^{1,1}(y,\theta) = \hat{F}^{1,1}(y,\theta) \exp[-\varepsilon^{2+\delta'}(y-x)^2], \qquad y\in\mathbb{R} \qquad (4.6)$$

where $\delta' \in (4/3, 2\delta)$ (see condition D$'$ above). The difference

$$\sum_{n\in\mathbb{Z}} 1/4\pi^2 \int_{-\pi}^{\pi} d\theta\, e^{-il\theta} \cos[\omega(\theta)\,\varepsilon^{-2}t]$$
$$\times \int_{-\pi}^{\pi} d\varphi\, e^{-in\varphi} \cos[\omega(\theta-\varphi)\,\varepsilon^{-2}t]$$
$$\times [\hat{F}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n),\theta) - \hat{F}_{(\varepsilon)}^{1,1}(\varepsilon([\varepsilon^{-1}x]-n),\theta)]$$

is estimated on the base of the same arguments as above. It is less than

$$c_8 \sum_{|n| > c_0 \varepsilon^{-2} t} \left| \int_{-\pi}^{\pi} d\varphi \, e^{-in\varphi} \cos[\omega(\theta - \varphi) \, \varepsilon^{-2} t] \right|$$

$$+ c_9 \sum_{0 \leqslant n \leqslant c_0 \varepsilon^{-2} t} \left| \int_{-\pi}^{\pi} d\varphi \, e^{-in\varphi} \cos[\omega(\theta - \varphi) \, \varepsilon^{-2} t] \right|$$

$$\times \{1 - \exp[-\varepsilon^{2+\delta'} (\varepsilon n - \varepsilon)^2]\}$$

$$\leqslant c_{10} \varepsilon^2 t + c_{11} \varepsilon^{2/3} \varepsilon^{-2} \varepsilon^{\delta'}$$

where $c_8, ..., c_{11}$ are positive constants. Hence, it is going to zero.

Now we can apply the Poisson summation formula. Denoting [cf. (3.5)]

$$\hat{F}^{1,1}_{\varepsilon,\xi}(-z, \theta) = \hat{F}^{1,1}_{(\varepsilon)}(\xi - z, \theta) \tag{4.7}$$

we get [cf. (3.5)]

$$\sum_n e^{in\varphi} \hat{F}^{1,1}_{(\varepsilon)}(\varepsilon([\varepsilon^{-1} x] - n), \theta) = \varepsilon^{-1} \sum_k \hat{\hat{F}}^{1,1}_{(\varepsilon), \varepsilon[\varepsilon^{-1} x]}(\varepsilon^{-1}(\varphi + 2k\pi), \theta) \tag{4.7'}$$

where $\hat{\hat{F}}^{1,1}_{(\varepsilon), \varepsilon[\varepsilon^{-1} x]}(v, \theta)$, $v \in \mathbb{R}^1$, $\theta \in [-\pi, \pi)$, denotes as before the Fourier transform of $\hat{F}^{1,1}_{(\varepsilon), \varepsilon[\varepsilon^{-1} x]}(-z, \theta)$ in the variable z. Substituting (4.7′) into the expression under investigation, we obtain

$$\sum_{n \in \mathbb{Z}} 1/4\pi^2 \int_{-\pi}^{\pi} d\theta \, e^{-il\theta} \cos[\omega(\theta) \, \varepsilon^{-2} t]$$

$$\times \int_{-\pi}^{\pi} d\varphi \, e^{-in\varphi} \cos[\omega(\theta - \varphi) \, \varepsilon^{-2} t] \, \hat{F}^{1,1}_{(\varepsilon)}(\varepsilon([\varepsilon^{-1} x] - n), \theta)$$

$$= 1/4\pi^2 \int_{-\pi}^{\pi} d\theta \, e^{-il\theta} \cos[\omega(\theta) \, \varepsilon^{-2} t]$$

$$\times \int_{-\pi}^{\pi} d\varphi \cos[\omega(\theta - \varphi) \varepsilon^{-2} t] \, \varepsilon^{-1} \sum_k \hat{\hat{F}}^{1,1}_{(\varepsilon), \varepsilon[\varepsilon^{-1} x]}(\varepsilon^{-1}(\varphi + 2k\pi), \theta) \tag{4.8}$$

The next step of the proof is to check that the contribution of the sum

$$\varepsilon^{-1} \sum_{k \neq 0} \hat{\hat{F}}^{1,1}_{\varepsilon, [\varepsilon^{-1} x]}(\varepsilon^{-1}(\varphi + 2k\pi), \theta)$$

goes to zero as $\varepsilon \to 0$. By using the representation (4.2), one writes

$$\hat{\hat{F}}^{1,1}_{(\varepsilon),\xi}(v, \theta) = (4\pi\varepsilon^{\delta' + 2})^{-1/2} \exp[iv(\xi - x)]$$

$$\times \int_{\mathbb{R}} \hat{\mu}^{1,1}(\theta, ds) \exp[-isx - (1/4) \, \varepsilon^{-\delta' - 2}(v + s)^2] \tag{4.9}$$

We have from (4.9)

$$\sum_{k\neq 0} |\,\hat{}\hat{F}^{1,1}_{(\varepsilon),\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}(\varphi+2k\pi),\theta)|$$

$$\leqslant \sum_{k\neq 0}(1/\sqrt{2\pi})\,\varepsilon^{-\delta'/2-1}\left|\int_{|s|<(\pi/2)\varepsilon^{-1}}\hat{\mu}^{1,1}(\theta,ds)\exp(isx)\right.$$
$$\left.\times\exp\{-1/2\varepsilon^{-\delta'-2}[\varepsilon^{-1}(\varphi+2k\pi)+s]^2\}\right|$$
$$+\sum_{k\neq 0}(1/\sqrt{2\pi})\,\varepsilon^{-\delta'/2-1}\left|\int_{|s|>(\pi/2)\varepsilon^{-1}}\hat{\mu}^{1,1}(\theta,ds)\exp(isx)\right.$$
$$\left.\times\exp\{-1/2\varepsilon^{-\delta'-2}[\varepsilon^{-1}(\varphi+2k\pi)+s]^2\}\right| \qquad (4.10)$$

The first sum on the rhs of (4.10) is not greater than

$$c_{12}\sup_\theta \operatorname{Var}(\hat{\mu}^{1,1}(\theta,\cdot))\,\varepsilon^{-\delta'/2-1}\left\{\sum_{k\geqslant 1}\exp[-1/2\varepsilon^{-\delta'-4}\pi^2(2k-3/2)^2]\right\}$$

(we use here the fact that, for $|s|\leqslant\varepsilon^{-1}\pi/2$, $|\varphi|\leqslant\pi$, and $k\geqslant 1$, the bound $|\varphi+2k\pi+\varepsilon s|\geqslant|2k-3/2|\,\pi$ holds), while the second one does not exceed

$$c_{13}\{\sup_\theta \operatorname{Var}(\hat{\mu}^{1,1}(\theta,\cdot)|_{\mathbb{R}\setminus(-\pi\varepsilon^{-1}/2,\pi\varepsilon^{-1}/2)})\}(1+\varepsilon^{-\delta'/2-1})$$

where we have done the summation

$$\sum_{k\neq 0}\exp\{-1/2\varepsilon^{-\delta'-2}[\varepsilon^{-1}(\varphi+2k\pi)+s]^2\}$$

under the integral and c_{12}, c_{13} are positive constants. Both expressions tend to zero as $\varepsilon\to 0$ due to condition D′ and the choice of δ'.

Therefore, as in the preceding section, we have to compute only the limit of the term with $k=0$ in the sum on the rhs of (4.8),

$$1/4\pi^2\int_{-\pi}^{\pi}d\theta\exp(-il\theta)\cos[\omega(\theta)\,\varepsilon^{-2}t]$$
$$\times\int_{-\pi}^{\pi}d\varphi\cos[\omega(\theta-\varphi)\,\varepsilon^{-2}t]\,\varepsilon^{-1}\,\hat{}\hat{F}^{1,1}_{\varepsilon,[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi,\theta)$$
$$=1/4\pi^2\int_{-\pi}^{\pi}d\theta\exp(-il\theta)\,\{\exp[i\omega(\theta)\,\varepsilon^{-2}t]+\exp[-i\omega(\theta)\,\varepsilon^{-2}t]\}/2$$
$$\times\int_{-\pi}^{\pi}d\varphi\{\exp[i\omega(\theta-\varphi)\,\varepsilon^{-2}t]$$
$$+\exp[-i\omega(\theta-\varphi)\,\varepsilon^{-2}t]\}/2\varepsilon^{-1}\,\hat{}\hat{F}^{1,1}_{\varepsilon,[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi,\theta) \qquad (4.11)$$

Writing the rhs of (4.11) as a four-term sum, we remark that, as in the preceding section, the contribution of those terms that contain the exponents with the same signs tends to zero. In fact, consider, for example, the integral

$$\int_{-\pi}^{\pi} d\theta \exp(-il\theta) \exp[i\omega(\theta)\, \varepsilon^{-2}t]$$
$$\times \int_{-\pi}^{\pi} d\varphi \exp[i\omega(\theta-\varphi)\, \varepsilon^{-2}t]\, \varepsilon^{-1}\, {}^{\wedge}\hat{F}^{1,1}_{(\varepsilon),\varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi, \theta)$$
$$= \int_{-\pi}^{\pi} d\theta \exp(-il\theta) \exp[i\omega(\theta)\, \varepsilon^{-2}t]$$
$$\times \int_{-\varepsilon^{-1}\pi}^{\varepsilon^{-1}\pi} d\varphi \exp[i\omega(\theta-\varepsilon\varphi)\, \varepsilon^{-2}t]\, {}^{\wedge}\hat{F}^{1,1}_{(\varepsilon),\varepsilon[\varepsilon^{-1}x]}(\varphi, \theta) \qquad (4.12)$$

Using the representation (4.9), it is not hard to prove that (4.12) tends to zero, because for any fixed $u > 0$, the limit of (4.12) coincides with the limit

$$\lim_{\varepsilon \to 0} \int_{-\pi}^{\pi} d\theta \exp(-il\theta) \exp[i\omega(\theta)\, \varepsilon^{-2}t]$$
$$\times \int_{-u}^{u} \hat{\mu}^{1,1}(\theta, ds) \exp(-isx) \exp[i\omega(\theta+\varepsilon\varphi)\, \varepsilon^{-2}t] = 0 \qquad (4.13)$$

In fact, let us substitute (4.9) in (4.12) and change the order of the integrals in $d\varphi$ and $\hat{\mu}^{1,1}(\theta, ds)$. The key remark is that the asymptotic behavior of the integral

$$(4\pi\varepsilon^{\delta'+2})^{-1/2} \int \hat{\mu}^{1,1}(\theta, ds)$$
$$\times \int_{-\varepsilon^{-1}\pi}^{\varepsilon^{-1}\pi} d\varphi \exp\{i[\varphi(\varepsilon[\varepsilon^{-1}x]-x) + \omega(\theta-\varepsilon\varphi)\, \varepsilon^{-2}t - sx]$$
$$-1/4\varepsilon^{-\delta'-2}(\varphi+s)^2\} \qquad (4.14)$$

is the same as that of

$$\int \hat{\mu}^{1,1}(\theta, ds) \exp\{i[\omega(\theta+\varepsilon s)\, \varepsilon^{-2}t - sx]\}$$

To prove this, we divide the interval $(-\varepsilon^{-1}\pi, \varepsilon^{-1}\pi)$ of the integration in $d\varphi$ into two parts, $\{\varphi: |\varphi| < \varepsilon^{-1}\pi, |\varphi + s| > \varepsilon^{1+\delta''}\}$ and $(-\varepsilon^{1+\delta''} - s)$, where $\delta'' \in (0, \delta'/2)$ is a constant. The contribution in (4.14) of the integral over the first set vanishes, while the asymptotic behavior of the second addend is not changed provided $\omega(\theta - \varepsilon\varphi)$ is replaced by $\omega(\theta + \varepsilon s)$. Finally, let us prove (4.13). Taking the three-term expansion of $\omega(\theta + \varepsilon s)$ in ε, we reduce the problem to proving the relation

$$\lim_{\varepsilon \to 0} \int_{-\pi}^{\pi} d\theta \exp(-il\theta) \exp[i2\omega(\theta)\, \varepsilon^{-2} t]$$
$$\times \int_{-u}^{u} \hat{\mu}^{1,1}(\theta, ds) \exp(-isx) \exp\{i[\omega'(\theta)\, \varepsilon^{-1} st + \omega''(\theta)\, s^2 t/2]\} = 0$$

which is easy to see, integrating by parts.

Therefore, it remains to evaluate those terms in (4.11) that contain the exponents with opposite signs. For definiteness, consider the integral

$$\int_{-\pi}^{\pi} d\theta \exp(-il\theta) \exp[-i\omega(\theta)\, \varepsilon^{-2} t]$$
$$\times \int_{-\pi}^{\pi} d\varphi \exp[i\omega(\theta - \varphi)\, \varepsilon^{-2} t]\, \varepsilon^{-1}\, \hat{\ } \hat{F}^{1,1}_{(\varepsilon), \varepsilon[\varepsilon^{-1}x]}(\varepsilon^{-1}\varphi, \theta)$$
$$= \int_{-\pi}^{\pi} d\theta \exp(-il\theta) \exp[-i\omega(\theta)\, \varepsilon^{-2} t]$$
$$\times \int_{-\varepsilon^{-1}\pi}^{\varepsilon^{-1}\pi} d\varphi\, (\exp[i\omega(\theta - \varepsilon\varphi)\, \varepsilon^{-2} t]\, \hat{\ } \hat{F}^{1,1}_{(\varepsilon), \varepsilon[e^{-1}x]}(\varphi, \theta) \qquad (4.15)$$

In the same way as before, one can establish that the asymptotics of (4.15) is the same as that of the integral

$$1/16\pi^2 \int_{-\pi}^{\pi} d\theta \exp(-il\theta) \int d\varphi \exp[-i\omega'(\theta)\, \varepsilon^{-1} \varphi t + i\omega''(\theta)\, \varphi^2 t/2]$$
$$\times\hat{\ } \hat{F}^{1,1}_{(\varepsilon), \varepsilon[\varepsilon^{-1}x]}(\varphi, \theta)$$
$$= 1/8\pi \int_{-\pi}^{\pi} d\theta\, [\exp(-il\theta)][i/2\pi\omega''(\theta)\, t]^{1/2}$$
$$\times \int dz \exp[-iz^2/2\omega''(\theta)\, t]$$
$$\times \hat{F}^{1,1}_{(\varepsilon)}(\varepsilon[\varepsilon^{-1}x] + \omega'(\theta)\, \varepsilon^{-1} t + z, \theta) \qquad (4.16)$$

Finally, returning to (4.6), it remains to check that the asymptotics is not changed when one replaces $\hat{F}^{1,1}_{(\varepsilon),\varepsilon[\varepsilon^{-1}x]}$ in (4.16) by $\hat{F}^{1,1}$. This may be easily done by integrating by parts.

In a similar way, one determines the asymptotics of all nonvanishing terms on the rhs of (4.11). This gives the (q–q) contribution into $\hat{F}^{1,1}_{(\varepsilon)}$. The other terms are treated similarly. This leads to the conclusion of Theorem 4.1.

ACKNOWLEDGMENTS

A. P. thanks the Italian CNR and the Academy of Sciences of the USSR for supporting his visit in Moscow. A. P. and Yu. M. S. are also grateful to Prof. S. Albeverio and Research Center BiBoS for their warm hospitality.

REFERENCES

1. R. L. Dobrushin, A. Pellegrinotti, Yu. M. Suhov, and L. Triolo, *J. Stat. Phys.* **43**:571–608 (1986).
2. A. De Masi, N. Ianiro, A. Pellegrinotti, and E. Presutti, in *Nonequilibrium Phenomena, II: From Stochastics to Hydrodynamics*, E. W. Montroll and J. L. Lebowitz, eds. (North-Holland, Amsterdam, 1984).
3. R. L. Dobrushin, Ya. G. Sinai, and Yu. M. Suhov, Dynamical system of statistical mechanics, in *Modern Problems of Mathematics*, Vol. 2 (VINITI, Moscow, 1985) [in Russian].
4. C. Boldrighini, R. L. Dobrushin, and Yu. M. Suhov, *J. Stat. Phys.* **31**:577–615 (1983).
5. C. Morrey, *Commun. Pure Appl. Math.* **8**:279–326 (1955).
6. O. E. Lanford and J. L. Lebowitz, *Lecture Notes in Physics*, No. 38 (Springer-Verlag, Berlin, 1975).
7. C. Boldrighini, A. Pellegrinotti, and L. Triolo, *J. Stat. Phys.* **30**:123–155 (1983).
8. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II (Wiley, New York, 1966).